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DEGENERATE PRINCIPAL SERIES AND INVARIANT DISTRIBUTIONS

BY

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ABSTRACT

In this article we give a description of the tempered distributions on a matrix space $M_{m,n}(\mathbf{R})$ which are invariant under the linear action of an orthogonal group $O(p,q)$, $p+q=m$. We also determine the points of reducibility of the degenerate principal series of the metaplectic group $\mathrm{Mp}(n,\mathbf{R})$ induced from a character of the maximal parabolic with $\mathrm{GL}(n,\mathbf{R})$ as Levi factor. Finally, we identify the representation of $\mathrm{MP}(n,\mathbf{R})$ which is associated to the trivial representation of $O(p,q)$ under the archimedean theta correspondence.

1. Introduction

In this note we will utilize some of the techniques of Guillemonat [4] to give a necessary and sufficient condition for the irreducibility of a certain degenerate principal series representation of $G=\mathrm{Mp}(n,\mathbf{R})$, the two-fold metaplectic cover of the symplectic group $\mathrm{Sp}(n,\mathbf{R})$. More precisely, as explained in the notation section below, we identify G as a set with $\mathrm{Sp}(n,\mathbf{R})\times\mu_2$. Then we consider the maximal parabolic

$$P=MN$$

with

$$M=\{(m(a),\varepsilon)\}$$

where, for $a\in\mathrm{GL}(n,\mathbf{R})$,

$$m(a)=\begin{pmatrix} a & \\ & {}^t a^{-1}\end{pmatrix}. \tag{0.1}$$

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Also

$$N = \{(n(b), 1)\}$$

where

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \tag{0.2}$$

and

$$b = {}^t b \in M_n(\mathbf{R}).$$

The group M has a character of order 4 given by:

$$\chi(m(a), \varepsilon) = \varepsilon \cdot \begin{cases} i & \text{if } \det a < 0, \\ 1 & \text{if } \det a > 0. \end{cases} \tag{0.3}$$

For $\alpha \in \mathbf{Z}/4\mathbf{Z}$, $s \in \mathbf{C}$, and $a \in \mathrm{GL}(n, \mathbf{R})$, let

$$\chi^\alpha(s)(m(a), \varepsilon) = |\det(a)|^s \chi(m(a), \varepsilon)^\alpha \tag{0.4}$$

and let

$$I^\alpha(s) = \mathrm{Ind}_P^G \chi^\alpha(s) \tag{0.5}$$

be the corresponding induced representation of G. Here the induction is normalized so that $I^\alpha(s)$ is unitary when $\mathrm{Re}(s) = 0$. These representations frequently play a role in the study of the theta correspondence for dual reductive pairs [17] [18] and in the construction of Rallis and Piatetski-Shapiro [13] [14] of integral representations of the standard Langlands L-functions.

THEOREM 1. *$I^\alpha(s)$ is reducible if and only if*

$$\begin{cases} s \in \rho_n + \alpha/2 + \mathbf{Z} & \text{if } n > 1, \\ s \in \mathbf{Z}, \text{ and } (-1)^s = -(-1)^{\alpha/2} & \text{if } n = 1 \text{ and } \alpha \text{ is even}, \\ s \in \frac{1}{2} + \mathbf{Z} & \text{if } n = 1 \text{ and } \alpha \text{ is odd}. \end{cases}$$

Here $\rho_n = (n+1)/2$.

The corresponding result in the p-adic case was proved by Gustafson [5].

Next we consider the dual reductive pair $(G, O(V))$ where V, $(\ , \ )$ is a nondegenerate inner product space over $\mathbf{R}$ of signature (p, q) where $m = p + q$. Let $\mathbf{S} = \mathbf{S}(V^n)$ be the Schwartz space of V^n, and let ω denote the action of G on $\mathbf{S}(V^n)$ via the oscillator representation. If we let $h \in O(V)$ act on $\varphi \in \mathbf{S}(V^n)$ by

$$\omega(h)\varphi(x) = \varphi(h^{-1}x) \tag{0.6}$$

as usual, then the delta function at the origin, δ_0, is an $O(V)$-invariant tempered distribution. Our second result is a description of all such distributions:

THEOREM 2. *The space of $O(V)$-invariant tempered distributions is the closure of the span of the G translates of δ_0, i.e.*

$$(\mathbf{S}(V^n)')^{O(V)} = \text{cl span}\{\omega(g)\delta_0 \mid g \in G\}.$$

In the p-adic case the analogous result was proved by Rallis [16, Theorem II.1.1]. For $n = 1$ this result is well known [11] [21]. For general n it has long been the expected result but, to our knowledge, had not been proved. Related questions have been considered in [15] [19] [20].

Theorem 2 is an easy consequence of a finer result. Let $S = S(V^n) \subset \mathbf{S} = \mathbf{S}(V^n)$ denote the space of Schwartz functions which correspond to polynomials in a Fock model of the oscillator representation. Then S is naturally a $(\mathfrak{g}, K) \times (\mathfrak{o}(V), L)$-module where $\mathfrak{g} = \text{Lie}(G)_{\mathbf{C}}$, $\mathfrak{o}(V) = \text{Lie}(O(V))_{\mathbf{C}}$, K is a maximal compact subgroup of G, and $L \simeq O(p) \times O(q)$. Let R be the maximal quotient of S which is trivial as an $(\mathfrak{o}(V), L)$-module. By Howe's result [6] R is a cyclic, quasi-simple $(\mathfrak{g}, K)$-module generated by the image of the Gaussian φ^0 and has a unique irreducible quotient Q. Note that Q is the representation of $(\mathfrak{g}, K)$ which corresponds to the trivial representation of $O(p, q)$ under Howe's quotient correspondence. On the other hand, the whole representation R might well be viewed as being associated to the trivial representation of $O(p, q)$, and thus it is of interest to determine its structure.

Let $s_0 = m/2 - (n+1)/2$ and let $\alpha = m + 2q \bmod 4$. Then the restriction to S of the map

$$\begin{aligned} \lambda : \mathbf{S}(V^n) &\to I^{\alpha}(s_0) \\ \varphi &\mapsto \omega(g)\varphi(0) \end{aligned} \tag{0.7}$$

factors through R.

THEOREM 3. *The map*

$$\lambda : R \to I^{\alpha}(s_0)$$

is injective.

Using Theorem 3 we may obtain information about when R is irreducible, give examples in which it has an interesting composition series, and obtain information about Q. For example, we can describe all cases in which Q is finite dimensional. Such information may also be found in [12].

As mentioned at the beginning, our main technique is that of [4]. In particular we exploit the scalar K-types in $I^\alpha(s)$ and the fact that all K-types have multiplicity one in this space. There is little doubt that Theorem 1 and perhaps our other results can be extended to the more general setting of groups of Hermitian tube type.

Notation

As explained above we let

$$G = \mathrm{Mp}(n, \mathbf{R})$$

be the metaplectic group and let

$$\mathfrak{g} = \mathrm{Lie}(G)_{\mathbf{C}} = \mathfrak{sp}(n, \mathbf{C}).$$

Let

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \middle| \, {}^tX_1 = -X_1, {}^tX_2 = X_2 \right\},$$

$$\mathfrak{p}_+ = \left\{ p_+(X) = \frac{1}{2}\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \middle| \, {}^tX = X \right\},$$

and $\mathfrak{p}_- = \overline{\mathfrak{p}_+}$. Then let K be the maximal compact subgroup of G with

$$\mathfrak{k} = \mathrm{Lie}(K)_{\mathbf{C}}$$

and note that we have a Harish-Chandra decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-.$$

For

$$d(x) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix},$$

let

$$h(x) = \begin{pmatrix} & -id(x) \\ id(x) & \end{pmatrix}$$

and let $\varepsilon_j(h(x)) = x_j$. Then $\mathfrak{h} = \{h(x) \mid x \in \mathbf{C}^n\}$ *is a Cartan subalgebra of* $\mathfrak{k}$, and a set of positive roots is given by $\Delta^+ = \Delta_c^+ \amalg \Delta_n^+$ where

$$\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leqq i \leqq j \leqq n\}$$

and

$$\Delta_c^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqq i < j \leqq n\}$$

are the non-compact and compact positive roots respectively. Note that $\gamma_j = 2\varepsilon_j$ with $j = 1, \ldots, n$ form a system of strongly orthogonal roots as in [4].

Let

$$e_j = \Big(0, \ldots, 0, \underset{j\text{-th}}{1}, 0, \ldots, 0\Big),$$

$$H_j = h(e_j),$$

and let

$$H = H_1 + \cdots + H_n = h(1, \ldots, 1). \tag{0.8}$$

Also let

$$X_j^+ = p_+(e_j), \qquad X_j^- = \overline{X_j^+}$$

and

$$E_j = X_j^+ + X_j^- = \begin{pmatrix} d(e_j) & \\ & -d(e_j) \end{pmatrix}.$$

Then let

$$E = E_1 + \cdots + E_n = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}. \tag{0.9}$$

Note that the stabilizer M_1 of $E \in \mathfrak{p}$ in K (which acts via Ad) has Lie algebra

$$\left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix} \,\middle|\, {}^tX_1 = -X_1 \right\},$$

and hence this group is the inverse image in G of $O(n) \subset U(n) \subset \mathrm{Sp}(n, \mathbf{R})$.

Let P be as in the introduction. If we identify G with $\mathrm{Sp}(n, \mathbf{R}) \times \mu_2$ in the standard way [16], then the multiplication in M is given by

$$(m(a_1), \varepsilon_1) \cdot (m(a_2), \varepsilon_2) = (m(a_1 a_2), \varepsilon_1 \varepsilon_2 (\det a_1, \det a_2)_{\mathbf{R}})$$

where $(\;,\;)_{\mathbf{R}}$ is the Hilbert symbol for $\mathbf{R}$. In particular, (0.4) indeed gives a character of M. Let

$$\mathfrak{P} = \mathrm{Lie}(P)_{\mathbf{C}}$$

$$\simeq \left\{ \begin{pmatrix} X & Y \\ 0 & -{}^tX \end{pmatrix} \middle| X \in M_n(\mathbf{C}) \text{ and } {}^tY = Y \in M_n(\mathbf{C}) \right\},$$

$$\mathfrak{m} = \mathrm{Lie}(M)_{\mathbf{C}}$$

$$\simeq \left\{ \begin{pmatrix} X & \\ & -{}^tX \end{pmatrix} \middle| X \in M_n(\mathbf{C}) \right\},$$

and

$$\mathfrak{n} = \mathrm{Lie}(N)_{\mathbf{C}}.$$

Also let

$$\mathfrak{k}' = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \in \mathfrak{k} \mid \mathrm{tr}(X_2) = 0 \right\}, \tag{0.10}$$

$$\mathfrak{m}' = \left\{ \begin{pmatrix} X & \\ & -{}^tX \end{pmatrix} \in \mathfrak{m} \mid \mathrm{tr}(X) = 0 \right\}, \tag{0.11}$$

and

$$\mathfrak{P}' = \mathfrak{m}' + \mathfrak{n} \tag{0.12}$$

Note that

$$\mathfrak{k} = \mathfrak{k}' + \mathbf{C} \cdot H$$

and

$$\mathfrak{m} = \mathfrak{m}' + \mathbf{C} \cdot E.$$

Finally, if $C \in U(\mathfrak{g})$ is the Casimir operator of $\mathfrak{g}$, then

$$C = C_{\mathfrak{k}} + C_{\mathfrak{p}} \tag{0.13}$$

where $C_{\mathfrak{k}}$ is the Casimir operator of $\mathfrak{k}$.

1. The criterion for irreducibility

In this section we use a slight variation of the technique of [4] to determine the main structural properties of $I^{\alpha}(s)$.

Let $M_1 \cong O(n) \times \mu_2$ be the inverse image of $O(n)$ in $K \cong U(n) \times \mu_2$, and note that, as a representation of K,

$$I^{\alpha}(s) \simeq \mathrm{Ind}_{M_1}^{K} \chi^{\alpha} \simeq (\mathrm{Ind}_{O(n)}^{U(n)} \mathbf{1}) \otimes \chi^{\alpha}. \tag{1.1}$$

In the last expression χ^α is the character of K whose differential on the maximal torus is given by the weight

$$\frac{\alpha}{2}(1, \ldots, 1). \tag{1.2}$$

Thus the K-types occurring in $I^\alpha(s)$ are precisely those whose twists by $(\chi^\alpha)^{-1}$ descend to $U(n)$ and contain the trivial representation of $O(n)$. By the result of [2] these are precisely the irreducible representations σ_λ of K whose highest weight

$$\lambda = (l_1, \ldots, l_n)$$

satisfy

$$l_i \in \frac{\alpha}{2} + 2\mathbf{Z} \tag{1.3}$$

for all i. Moreover, these K-types occur with multiplicity one. In particular, for $l \in \frac{1}{2}\alpha + 2\mathbf{Z}$, we define $\Phi^l(s) \in I^\alpha(s)$ by

$$\Phi^l(s)(k) = (\det k)^l \tag{1.4}$$

for $k \in K$. Here, of course $k \mapsto (\det k)^l$ is the character of K with weight $(l, \ldots, l)$.

Let $\mathbf{J} \subset U(\mathfrak{g})$ be the left ideal generated by all the K-isotypic subspaces $U(\mathfrak{g})_\sigma$ where $\sigma\,\big|_{M_1}$ does not contain the trivial representation. Note that $\mathbf{J}$ will then annihilate $\Phi^l(s)$ for any l.

One of the main results of [4] is an explicit description of the complement of $\mathbf{J}$ in $U(\mathfrak{g})$. For any

$$\mu = 2(r_1, \ldots, r_n) \tag{1.5}$$

with $r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq 0$, let

$$u_\mu^0 \in S(\mathfrak{p}_+)_{|\mu|}$$

be a non-zero highest weight vector of weight μ. Here $|\mu| = r_1 + \cdots + r_n$ is the degree of the symmetric tensor. Similarly, if

$$\nu = -2(s_1, \ldots, s_n) \tag{1.6}$$

with $0 \leqq s_1 \leqq \cdots \leqq s_n$, let

$$v_\nu^0 \in S(\mathfrak{p}_-)_{|\nu|}$$

be a non-zero highest weight vector of weight ν.

Let

$$\iota_\pm : S(\mathfrak{p}_\pm) \simeq U(\mathfrak{p}_\pm) \hookrightarrow U(\mathfrak{g}) \tag{1.7}$$

be the natural homomorphism of $S(\mathfrak{p}_\pm)$ into $U(\mathfrak{g})$ induced by the inclusion $\iota_\pm : \mathfrak{p}_\pm \hookrightarrow \mathfrak{g}$, and let

$$u_\mu = \iota_+(u_\mu^0) \tag{1.8}$$

and

$$v_\nu = \iota_-(v_\nu^0). \tag{1.9}$$

Finally, let H and $C_\mathfrak{p}$ be given by (0.8) and (0.13), and let $\mathscr{E}$ be the K subspace of $U(\mathfrak{g})$ generated by the highest weight vectors

$$u_\mu v_\nu p(H, C_\mathfrak{p}) \tag{1.10}$$

where μ and ν are disjoint, i.e., there is an index t_0 such that $r_t = 0$ for $t > t_0$ and $s_t = 0$ for $t \leqq t_0$. Also $p(H, C_\mathfrak{p})$ is a polynomial in H and $C_\mathfrak{p}$. Then [4, Theorem p. 103] asserts that

$$U(\mathfrak{g}) = \mathscr{E} \oplus \mathbf{J}. \tag{1.11}$$

Thus, for any l,

$$U(\mathfrak{g}) \cdot \Phi^l(s) = \mathscr{E} \cdot \Phi^l(s). \tag{1.12}$$

We will need the following fact:

Proposition 1.1. *Suppose that $\lambda = (l_1, \ldots, l_n)$ is a highest weight satisfying (1.3), as above, and let $\Phi^\lambda(s) \in I^\alpha(s)$ be a non-zero highest weight vector of weight λ. Then*

$$\Phi^\lambda(s)(e) \neq 0.$$

Proof. By restriction to K we have

$$I^\alpha(s) \simeq \operatorname{Ind}_{M_1}^K \chi^\alpha \simeq \operatorname{Ind}_{O(n)}^{U(n)} \mathbf{1} \tag{1.13}$$

where the second isomorphism is given by multiplication by the character χ^α of K. Now we have an inclusion

$$\begin{aligned} O(n) \backslash U(n) &\hookrightarrow \mathrm{Sym}_n(\mathbf{C}) \\ k &\mapsto {}^t k k \end{aligned} \tag{1.14}$$

which is equivariant with respect to the right action of $U(n)$ on both sides. Let X be the algebraic variety defined over $\mathbf{R}$ given by $X = R_{\mathbf{C}/\mathbf{R}}\,\mathrm{Sym}(n)$ where $R_{\mathbf{C}/\mathbf{R}}$ is Weil's restriction of scalars. Then $X(\mathbf{R}) \simeq \mathrm{Sym}_n(\mathbf{C})$ and $X(\mathbf{C}) \simeq \mathrm{Sym}_n(\mathbf{C}) \times \mathrm{Sym}_n(\mathbf{C})$ with the automorphism σ of $\mathbf{C}/\mathbf{R}$ acting by

$$\sigma : (x_1, x_2) \mapsto (\bar{x}_2, \bar{x}_1). \tag{1.15}$$

Also let G be the algebraic group over $\mathbf{R}$ such that $G(\mathbf{R}) = U(n)$, and hence $G(\mathbf{C}) \simeq \mathrm{GL}(n, \mathbf{C})$ with σ acting by $\sigma(g) = \overline{({}^tg^{-1})}$. If we let $g \in G(\mathbf{C})$ act on $X(\mathbf{C})$ on the right by

$$(x_1, x_2)g = ({}^tgx_1g,\, g^{-1}x_2{}^tg^{-1}), \tag{1.16}$$

then the action of $U(n)$ on $\mathrm{Sym}_n(\mathbf{C})$ above is just the action of $G(\mathbf{R})$ on $X(\mathbf{R})$. Let $\mathscr{R}$ be the ring of regular functions on X. Then among the highest weight vectors for the action of $G(\mathbf{C}) = \mathrm{GL}(n, \mathbf{C})$ on $\mathscr{R}$ are the monomials of the form

$$\delta_\lambda = (\delta_1^+)^{r_1} \cdots (\delta_t^+)^{r_t}(\delta_{t+1}^-)^{s_{t+1}} \cdots (\delta_n^-)^{s_n}, \tag{1.17}$$

where δ_j^+ is the upper principal $j \times j$ minor of x_1 and δ_j^- is the lower principal $(n-j+1) \times (n-j+1)$ minor of x_2. This monomial has weight λ given by

$$2(r_1 + \cdots + r_t, r_2 + \cdots + r_t, \ldots, r_t, -s_{t+1}, -s_{t+1} - s_{t+2}, \ldots, -s_{t+1} - \cdots - s_n). \tag{1.18}$$

But now the pullbacks of these functions to $O(n) \backslash U(n)$, where

$$O(n) \backslash U(n) \hookrightarrow X(\mathbf{R}) \subset X(\mathbf{C}),$$

are precisely the non-zero highest weight vectors in $\mathrm{Ind}_{O(n)}^{U(n)}\,\mathbf{1}$. It is clear from the construction that the values of these functions at $e \in K$ are non-zero. ■

Using Proposition 1.1, we henceforth normalize the highest weight vector Φ^λ so that $\Phi^\lambda(e) = 1$.

Next observe that if $X \in U(\mathfrak{g})$ is a highest weight vector of weight λ for K, then

$$X \cdot \Phi^l(s) = c(l, \lambda, X, s)\Phi^{l+\lambda}(s) \tag{1.19}$$

where, by Proposition 1.1,

$$c(l, \lambda, X, s) = X \cdot \Phi^l(s)(e). \tag{1.20}$$

For E given by (0.9) and H given by (0.8) we let

$$\xi : U(\mathfrak{g}) \rightarrow \mathbf{C}[E] \otimes \mathbf{C}[H] \tag{1.21}$$

be the projection induced by the decomposition

$$U(\mathfrak{g}) = \mathfrak{P}'U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}' + \mathbf{C}[E]\otimes\mathbf{C}[H], \tag{1.22}$$

where $\mathfrak{P}'$ and $\mathfrak{k}'$ are given by (0.12) and (0.10) respectively. Then it is easy to check that, for any $X \in U(\mathfrak{g})$,

$$X\cdot\Phi^l(s)(e) = \xi(X)\Phi^l(s)(e). \tag{1.23}$$

Thus we need to know the quantities $\xi(u_\mu v_\nu)$ for μ and ν as above. Guillemonat proved [4, Theorem, top of p. 112] that

$$\xi(u_\mu) = (-2)^{-|\mu|}\prod_{k=1}^{n}\prod_{j=0}^{r_k-1}\left(\frac{1}{n}(E+H) - k + 1 + 2j\right), \tag{1.24}$$

where $|\mu| = r_1 + \cdots + r_n$, and

$$\xi(v_\nu) = 2^{-|\nu|}\prod_{k=1}^{n}\prod_{j=0}^{s_k-1}\left(\frac{1}{n}(E+H) - n + k + 2j\right). \tag{1.25}$$

The following slight extension of the results of [3] will be proved in Section 4 below:

PROPOSITOIN 1.2. *For μ and ν disjoint as above,*

$$\xi(u_\mu v_\nu) = \xi(u_\mu)\xi(v_\nu).$$

Finally, an easy computation shows:

LEMMA 1.3. (i) $H\cdot\Phi^l(s) = nl\Phi^l(s)$.

(ii) $E\cdot\Phi^l(s)(e) = n(s+\rho_n)$.

(iii) $C_p\cdot\Phi^l(s) = *\Phi^l(s)$ *where* $*$ *is a scalar (which plays no role in what follows).*

Now let

$$P^l_\mu(s) = \prod_{k=1}^{n}\prod_{j=0}^{r_k-1}(s+\rho_n+l-k+1+2j) \tag{1.26}$$

and

$$Q^l_\nu(s) = \prod_{k=1}^{n}\prod_{j=0}^{s_k-1}(s+\rho_n-l-n+k+2j). \tag{1.27}$$

Combining the above facts we obtain:

COROLLARY 1.4. *For μ and ν disjoint as above*

$$u_\mu v_\nu \Phi^l(s) = cP^l_\mu(s)Q^l_\nu(s)\Phi^\lambda(s)$$

where $\lambda = \mu + \nu + (l, \ldots, l)$ *and* c *is a non-zero constant.*

Clearly Corollary 1.4 gives a necessary and sufficient condition — which we do not make explicit — for $\Phi^l(s)$ to be a cyclic vector for $I^\alpha(s)$.

We would like to determine when $\Phi^l(s)$ and $\Phi^\lambda(s)$ generate the same submodule of $I^\alpha(s)$. To do this consider the non-degenerate pairing

$$\langle \mid \rangle : I^\alpha(s) \times I^\alpha(-\bar{s}) \to \mathbf{C}$$

given by

$$\langle f_1 \mid f_2 \rangle = \int_{P\backslash G} f_1(g)\overline{f_2(g)}\,dg. \tag{1.28}$$

Let $\sigma : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the involution given by $\sigma(X) = -\bar{X}$ for $X \in \mathfrak{g}$. Then for $X \in U(\mathfrak{g})$ the $\Phi^l(s)$ component of $X \cdot \Phi^\lambda(s)$ is non-zero if and only if

$$\langle X\Phi^\lambda(s) \mid \Phi^l(-\bar{s})\rangle = \langle \Phi^\lambda(s) \mid \sigma(X)\Phi^l(-\bar{s})\rangle \neq 0. \tag{1.29}$$

By (1.10), (1.12) and Lemma 1.3 we may assume that $\sigma(X) = u_\mu v_\nu$, and hence we obtain:

Proposition 1.5. *For* μ *and* ν *disjoint and for*

$$\lambda = \mu + \nu + (l, \ldots, l),$$

$\Phi^l(s)$ *and* $\Phi^\lambda(s)$ *generate the same submodule of* $I^\alpha(s)$ *if and only if*

$$P^l_\mu(s)P^l_\mu(-s)Q^l_\nu(s)Q^l_\nu(-s) \neq 0.$$

Note that this result is analogous to the combination of the Lemma at the bottom of page 107 and the Theorem at the bottom of page 112 in [3].

Corollary 1.6. $I^\alpha(s)$ *is irreducible if and only if*

$$P^l_\mu(s)P^l_\mu(-s)Q^l_\nu(s)Q^l_\nu(-s) \neq 0$$

for all disjoint μ *and* ν.

If we take some $l \in \alpha/2 + 2\mathbf{Z}$, then the points of reducibility are given by

$$\pm s = \begin{cases} \rho_n + l - k + 1 + 2j \\ \rho_n - l - n + k + 2j \end{cases} \tag{1.30}$$

where $j \in \mathbf{Z}_{\geqq 0}$. A little algebra then yields the points of reducibility given in Theorem 1.

2. Howe's quotient R

In this section we will prove Theorem 3. This result will, in turn, be used in Section 3 to derive the description of invariant distributions given in Theorem 2.

As in the introduction we consider the $(\mathfrak{g}, K) \times (\mathfrak{o}(V), L)$-module S and its quotient R, which is the maximal quotient on which $(\mathfrak{o}(V), L)$ acts trivially. We begin by considering the K-spectrum of R.

PROPOSITION 2.1. *The K-types in R are those with highest weights*

$$\lambda = (l, \dots, l) + (a_1, \dots, a_r, 0, \dots, 0, -b_s, \dots, -b_1)$$

where $l = (p-q)/2$ and $a_1 \geqq \cdots \geqq a_r > 0$ and $b_1 \geqq \cdots \geqq b_s > 0$ for even integers a_i and b_j with $r \leqq p$ and $s \leqq q$. Moreover each such K-type occurs with multiplicity at most one.

PROOF. We consider the seesaw dual pair

$$\begin{array}{ccc} \mathrm{Sp}(n, \mathbf{R}) & & U(p,q) \\ \uparrow & \searrow\!\!\!\!\nearrow & \uparrow \\ U(n) & & O(p,q), \end{array} \tag{2.1}$$

and suppose that σ is an (irreducible) K-type occurring in R. Let S_σ be the σ-isotypic submodule of S so that, by the standard result of Howe [7] or Kashiwara and Vergne [8],

$$S_\sigma \simeq \sigma \otimes \pi(\sigma) \tag{2.2}$$

for some irreducible $(\mathfrak{u}(p,q), K')$-module, where K' is the inverse image of $U(p) \times U(q)$ in the ambient metaplectic group. Moreover, we know that if we let $\mathfrak{g}' = \mathfrak{u}(p,q)_{\mathbf{C}}$ and consider the Harish-Chandra decomposition

$$\mathfrak{g}' = \mathfrak{p}'_+ + \mathfrak{p}'_- + \mathfrak{k}', \tag{2.3}$$

then there is an irreducible representation τ of K' occurring in $\pi(\sigma)$ with multiplicity one such that

$$\pi(\sigma) = U(\mathfrak{g}') \cdot \pi(\sigma)_\tau = U(\mathfrak{p}'_+) \cdot \pi(\sigma)_\tau, \tag{2.4}$$

and hence

$$S_\sigma = U(\mathfrak{g}')\cdot S_{\sigma,\tau} = U(\mathfrak{p}'_+)\cdot S_{\sigma,\tau} \tag{2.5}$$

where $S_{\sigma,\tau} \subset S_\sigma$ is the τ-isotypic subspace. On the other hand, if we let $\mathfrak{g}'' = \mathfrak{o}(p, q)_{\mathbf{C}} = \mathfrak{p}'' + \mathfrak{k}''$ be the Cartan decomposition compatible with that of $\mathfrak{g}'$ above, then by [6]

$$\mathfrak{p}'_+ + \mathfrak{p}'_- = \mathfrak{p}'_+ + \mathfrak{p}'' = \mathfrak{p}'' + \mathfrak{p}'_-, \tag{2.6}$$

and this yields

LEMMA 2.2.

$$S_\sigma = U(\mathfrak{g}'')\cdot S_{\sigma,\tau}.$$

This is a special case of Proposition 3.1 of [6].

As a consequence, since $\mathfrak{g}''$ acts trivially on R, we see that:

LEMMA 2.3.

$$R_\sigma = \mathrm{pr}(S_\sigma) = \mathrm{pr}(S_{\sigma,\tau}),$$

where $\mathrm{pr} : S \to R$ *is the natural quotient map.*

But now the projection $\mathrm{pr} : S_{\sigma,\tau} \to R$ factors through the projection

$$V_\sigma \otimes V_\tau \to V_\sigma \otimes (V_\tau)^{O(p)\times O(q)} \tag{2.7}$$

where we temporarily write (σ, V_σ) and (τ, V_τ) for the irreducible representations of K and K'. Here recall that by (0.7) the action of $O(p, q) \supset O(p) \times O(q)$ in S is the natural linear action.

Thus we conclude that if the K-type σ is to occur in R, then the K'-type $\tau = \tau(\sigma)$ which corresponds to σ must contain an $O(p) \times O(q)$ invariant for the *linear action* of this group. But it is shown in [8] that $S_\sigma \neq 0$ implies that σ has weight λ as in the Proposition, but without the parity condition on the a_i's and b_j's. Moreover, τ then has highest weight

$$\mu = \left(a_1 + \frac{n}{2}, \ldots, a_r + \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2}; -\frac{n}{2}, \ldots, -\frac{n}{2}, -b_s - \frac{n}{2}, \ldots, -b_1 - \frac{n}{2}\right), \tag{2.8}$$

for K'.

Let K'' denote the inverse image of $O(p) \times O(q)$ in the metaplectic group, so that $K'' \subset K'$. The linear action of $O(p) \times O(q)$ in S differs from the action of K'' by a character. To determine the effect of this shift we note that the

Gaussian φ^0, which is an $O(p)\times O(q)$ invariant for the linear action, corresponds to the case

$$\lambda_0=(l,\ldots,l)\quad\text{and}\quad \mu_0=\left(\frac{n}{2},\ldots,\frac{n}{2};-\frac{n}{2},\ldots,-\frac{n}{2}\right).$$

Thus, in general, the restriction of the representation $\tau(\sigma)\otimes\mu_0^{-1}$ of highest weight

$$\mu-\mu_0=(a_1,\ldots,a_r,0,\ldots,0,-b_s,\ldots,-b_1) \tag{2.9}$$

to K'' descends to $O(p)\times O(q)$ and yields the linear action of this group in $S_{\sigma,\tau}$. Here again, by [2], we see that $\tau=\tau(\sigma)$ will contain an $O(p)\times O(q)$ invariant if and only if the a_i's and b_j's are even integers. This proves the first part of the Proposition.

Finally we note that since

$$\dim V_\tau^{O(p)\times O(q)}\leqq 1, \tag{2.10}$$

Lemma 2.3 shows that the multiplicity of σ in R is at most one.

Note that Propositon 2.1 is a slight refinement of the result of [9] [10].

Corollary 2.4. **J** *annihilates the image of the Gaussian* φ^0 *in* R.

To complete the proof of Theorem 3 we must show that the K-spectrum of R coincides with that of the cyclic submodule of $I^\alpha(s_0)$ generated by $\Phi^l(s_0)$ where $l=(p-q)/2$.

For λ as in Proposition 2.1 write $\lambda=(l,\ldots,l)+\mu+\nu$ where

$$\mu=2(a_1,\ldots,a_r,0,\ldots,0)\quad\text{and}\quad \nu=2(0,\ldots,0,-b_s,\ldots,-b_1)$$

are disjoint. Note that we have factored out a 2 and shifted the notation. Then by Corollary 1.4 and the decomposition (1.11), the vector $\Phi^\lambda(s_0)$ lies in the image of R, i.e., in the cyclic submodule generated by $\Phi^l(s_0)$, if and only if

$$P_\mu^l(s_0)Q_\nu^l(s_0)\neq 0. \tag{2.11}$$

But this factor is just a non-zero multiple of

$$\prod_{k=1}^{r}\prod_{j=0}^{a_k-1}(p-k+1+2j)\prod_{k=1}^{s}\prod_{j=0}^{b_k-1}(q-k+1+2j), \tag{2.12}$$

and is clearly non-zero if $p\geqq r$ and $q\geqq s$. Hence all of the possible K-types described in Proposition 2.1 actually occur in the image of R. This proves Theorem 3 and also yields:

COROLLARY 2.5. *The K-types described in Proposition 2.1 all occur in R with multiplicity 1.*

COROLLARY 2.6. *Let Q be the unique irreducible quotient of R as in the introduction. Then Q is finite dimensional if and only if*

$$n + 1 \leqq \min(p, q)$$

and

$$n + 1 \equiv p \equiv q \bmod 2.$$

Moreover, in this case, Q is the pullback to G of the finite dimensional representation of $\mathrm{Sp}(n, \mathbf{R})$ *with highest weight*

$$\left(\frac{m}{2} - n - 1, \ldots, \frac{m}{2} - n - 1\right).$$

PROOF. The K-types which occur in Q are precisely those of highest weight λ for which $\Phi^\lambda(s_0)$ and $\Phi^l(s_0)$ generate the same cyclic submodule of $I^\alpha(s_0)$, and these are characterized by Proposition 1.5. Now Q is finite dimensional if and only if there are only a finite number of such K-types, and this occurs precisely when the non-vanishing of

$$P^l_\mu(s_0)Q^l_\nu(s_0)P^l_\mu(-s_0)Q^l_\nu(-s_0)$$

imposes an upper bound on a_1 and b_1 in Proposition 2.1. It is then easy to check that such a bound is imposed precisely when

$$p - n - 1 = 2x \quad \text{and} \quad q - n - 1 = 2y$$

for $x, y \in \mathbf{Z}_{\geqq 0}$, in which case $a_1 \leqq y$ and $b_1 \leqq x$. It is then immediate that the highest weight of Q is as claimed.

Finally we note

COROLLARY 2.7. *R is irreducible in precisely the following cases:*
(a) $pq = 0$,
(b) $pq > 0$, $n = 1$, *and* p, q *odd*,
(c) $p, q, n > 1$ *and* $n + 1 \geqq m$,

$$\text{(d)}\quad \left\{\begin{matrix} p=1, q>0, n>1 \\ \textit{or} \\ q=1, p>0, n>1 \end{matrix}\right\} \quad \textit{and} \quad \left\{\begin{matrix} n+1 \geqq m \\ \textit{or} \\ m-n-1>0 \textit{ and even} \end{matrix}\right\}.$$

3. Invariant distributions

In this section we derive the description of the $O(p, q)$-invariant distributions given in Theorem 2.

Let $\mathbf{I} = I^\alpha_\infty(s_0)$ be the space of smooth vectors in $I^\alpha(s_0)$, and consider the map

$$\lambda : \mathbf{S} = \mathbf{S}(V^n) \rightarrow \mathbf{I} = I^\alpha_\infty(s_0), \tag{3.1}$$

which is a continuous map of locally convex topological vector spaces. Here, as in (1.13), $\mathbf{I}$ is identified with $C^\infty(O(n)\setminus U(n))$ and is given the corresponding Frechet topology. Let $\mathbf{R} = \lambda(\mathbf{S})$ be the image of $\mathbf{S}$ in $\mathbf{I}$.

PROPOSITION 3.1. *The space* $\mathbf{R} = \lambda(\mathbf{S})$ *is closed in* $\mathbf{I}$.

PROOF. We apply a result of Casselman [3]. First note that the representations of G on the Frechet spaces $\mathbf{S}$ and $\mathbf{I}$ are of moderate growth in the sense of [3]. Thus, since the underlying Harish–Chandra module R of $\mathbf{R} = \lambda(\mathbf{S})$ has finite length, the image $\lambda(\mathbf{S})$ is closed, [3, Cor. 10.5].

We let $\mathbf{S}'$, $\mathbf{R}'$, $\mathbf{I}'$, etc. denote the continuous duals of $\mathbf{S}$, $\mathbf{R}$, $\mathbf{I}$, etc. Recall that

$$(\ker(\lambda))^\perp = \text{weak closure of } \lambda'(\mathbf{R}')$$

[22, Prop. 35.4, p. 364]. This fact together with Banach's theorem [22, Theorem 32.7] now yield:

COROLLARY 3.2. (i) $\lambda' : \mathbf{R}' \rightarrow \mathbf{S}'$ *is continuous and injective.*

(ii) *Moreover,* $\lambda'(\mathbf{R}')$ *is weakly closed in* $\mathbf{S}'$, *and hence.*

(iii) $(\ker(\lambda))^\perp = \lambda'(\mathbf{R}')$.

Next we utilize our algebraic results.

PROPOSITION 3.3. *Suppose that*

$$T \in (\mathbf{S}')^{O(p,q)}$$

is an invariant distribution. Then

$$T \in (\ker(\lambda))^\perp.$$

PROOF. We begin by choosing a nice orthonormal bais for $L^2(V^n)$.

LEMMA 3.4. *There exists an orthonormal basis* $\{\varphi_\alpha\}_{\alpha\in A}$ *for* $L^2(V^n)$ *such that*
(i) $\varphi_\alpha \in S$ *for all* α,
(ii) *there exists a decomposition* $A = A_0 \amalg A_1$ *such that*

$$\lambda(\varphi_\alpha) = 0 \Leftrightarrow \alpha \in A_0$$

and the vectors $\lambda(\varphi_\alpha)$ *for* $\alpha \in A_1$ *are orthogonal with respect to the norm on* $\mathbf{I}$ *given by integration over* $O(n)\backslash U(n)$.

PROOF. For $\sigma \in \hat{K}$ let $S_\sigma \supset S_{\sigma,\tau}$ be as in Section 2. We then let

$$S_\sigma^1 = S_{\sigma,\tau}^{O(p)\times O(q)}, \tag{3.3}$$

and let

$$S_\sigma^0 = (S_\sigma^1)^\perp \tag{3.4}$$

be the orthogonal complement of S_σ^1 in S_σ. These spaces are K-invariant and we have a direct sum decomposition

$$S = \bigoplus_\sigma (S_\sigma^1 \oplus S_\sigma^0) \tag{3.5}$$

which is orthogonal for the $L^2(V^n)$ inner product. Thus we may choose our orthonormal basis $\{\varphi_\alpha\}_{\alpha\in A}$ compatible with this decomposition, and we set

$$A_0 = \{\alpha \in A \mid \varphi_\alpha \in S_\sigma^0 \text{ for some } \sigma\} \tag{3.6}$$

and

$$A_1 = \{\alpha \in A \mid \varphi_\alpha \in S_\sigma^1 \text{ for some } \sigma\}. \tag{3.7}$$

By Lemma 2.3 and the discussion following it we know that

$$\ker(\lambda) \cap S = \bigoplus_\sigma (S_\sigma^0), \tag{3.8}$$

and so we have proved the lemma. ■

Now suppose that $\varphi \in \ker(\lambda)$ and write

$$\varphi = \sum_{\alpha\in A} a_\alpha \varphi_\alpha$$

in $L^2(V^n)$. For $\alpha \in A$ with $\varphi_\alpha \in S_\sigma$ let

$$|\alpha|^2 = |\sigma|^2 = \sum_i l_j^2$$

if σ has highest weight $(l_1, \ldots, l_n)$. Then by [1] the sequence

$$\varphi_N = \sum_{\substack{\alpha \\ |\alpha| \leq N}} a_\alpha \varphi_\alpha$$

converges to φ in **S**, and by the continuity of λ

$$0 = \lim_{N\to\infty} \lambda(\varphi_N) = \lim_{N\to\infty} \left(\sum_{\substack{\alpha \\ |\alpha| \leq N \\ \alpha\in A_1}} a_\alpha \lambda(\varphi_\alpha)\right).$$

Since the $\lambda(\varphi_\alpha)$'s for $\alpha \in A_1$ are linearly independent in **I**, we must have $a_\alpha = 0$ for all $\alpha \in A_1$, and hence $\varphi_N \in \ker(\lambda)$ for all N. Thus

$$T(\varphi) = \lim_{N\to\infty} T(\varphi_N) = \lim_{N\to\infty} (\xi_0 \circ \lambda(\varphi_N)) = 0.$$

Here we have used Theorem 3 to conclude that the restriction of T to S factors through the restriction of λ to S, and ξ_0 is the corresponding linear functional on R. This proves Proposition 3.3. ■

Finally we note that since $\mathbf{R} = \lambda(\mathbf{S})$ is closed in **I**, the restriction map $\mathbf{I}' \to \mathbf{R}'$ is surjective [22, Proposition 35.5]. Thus we have proved

Proposition 3.5. *For every $T \in (\mathbf{S}')^{O(p,q)}$ there exists $\xi \in \mathbf{I}'$ such that*

$$T(\varphi) = \xi(\lambda(\varphi))$$

for all $\varphi \in \mathbf{S}$.

Finally we note that $\mathbf{I}'$ is the weak closure of the span of the delta functions δ_g for $g \in G$. Since

$$\lambda'(\delta_g) = \omega(g)\delta_0, \tag{3.9}$$

these distributions are again weakly dense in $(\mathbf{S}')^{O(p,q)}$, and this is precisely the assertion of Theorem 2.

Remark. By duality the existence of a unique irreducible quotient of R implies that there is a unique irreducible subspace of $(\mathbf{S}')^{O(p,q)}$ under the action of G. This irreducible subspace is finite dimensional if and only if the conditions of Corollary 2.6 are satisfied. In that case, we conclude that the tempered distribution T given by

$$T(\varphi) = \int_K \omega(k)\varphi(0)(\det k)^{n+1-m/2} dk$$

is a highest weight vector of this submodule. It would be interesting to give a more intrinsic description of these distributions like that given in [21] in the case $n = 1$.

4. Proof that $\xi(u_\mu v_\nu) = \xi(u_\mu)\xi(v_\nu)$

In this section we give the proof of Proposition 1.2.

Suppose that

$$\mu = 2(r_1, \dots, r_t, 0, \dots, 0) \quad \text{and} \quad \nu = -2(0, \dots, 0, s_{t+1}, \dots, s_n)$$

are disjoint as before. Let $\mathfrak{g}_t$ be the subalgebra generated by

$$\{X_\alpha \mid \alpha = \pm(\varepsilon_i \pm \varepsilon_j) \text{ with } 1 \leqq i \leqq j \leqq t\} \tag{4.1}$$

and let $\mathfrak{g}_t'$ be the subalgebra generated by

$$\{X_\alpha \mid \alpha = \pm(\varepsilon_i \pm \varepsilon_j) \text{ with } t < i \leqq j \leqq n\}. \tag{4.2}$$

Here X_α is a basis vector for the root space $\mathfrak{g}_\alpha$. Then $\mathfrak{g}_t \simeq \mathfrak{sp}(t, \mathbf{C})$ and $\mathfrak{g}_t' \simeq \mathfrak{sp}(n-t, \mathbf{C})$. We may define projections

$$\xi_t : U(\mathfrak{g}_t) \to \mathbf{C}[E_t] \otimes \mathbf{C}[H_t] \tag{4.3}$$

and

$$\xi_t' : U(\mathfrak{g}_t') \to \mathbf{C}[E_t'] \otimes \mathbf{C}[H_t'] \tag{4.4}$$

analogous to ξ. Note that

$$E_t - \frac{t}{n} E \in \mathfrak{m}', \quad E_t' - \frac{n-t}{n} E \in \mathfrak{m}', \quad H_t - \frac{t}{n} H \in \mathfrak{k}',$$

and

$$H_t' - \frac{n-t}{n} H \in \mathfrak{k}'. \tag{4.5}$$

Moreover, it is easy to check that if we view $U(\mathfrak{g}_t)$ and $U(\mathfrak{g}_t')$ as subalgebras of $U(\mathfrak{g})$, then

$$u_\mu \in U(\mathfrak{g}_t) \tag{4.6}$$

and

$$v_\nu \in U(\mathfrak{g}_t'). \tag{4.7}$$

Observe that $U(\mathfrak{g}_t)$ and $U(\mathfrak{g}_t')$ centralize each other in $U(\mathfrak{g})$.

Now write

$$u_\mu = \xi_t(u_\mu) + a_\mu + b_\mu \tag{4.8}$$

and

$$v_\nu = \xi_t'(v_\nu) + a_\nu + b_\nu, \tag{4.9}$$

with $a_\mu \in \mathfrak{P}_t' U(\mathfrak{g}_t)$, $b_\mu \in \mathbf{C}[E_t]U(\mathfrak{k}_t)\mathfrak{k}_t'$ etc. Note that $a_\mu \in \mathfrak{P}' U(\mathfrak{g})$ and $b_\mu \in U(\mathfrak{g})\mathfrak{k}'$ as well. Then the product

$$\begin{aligned} u_\mu v_\nu &= (\xi_t(u_\mu) + a_\mu + b_\mu)(\xi_t'(v_\nu) + a_\nu + b_\nu) \\ &\equiv \xi_t(u_\mu)\xi_t'(v_\nu) \bmod \mathfrak{P}'U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}', \end{aligned} \tag{4.10}$$

since we may pull a_μ or a_ν to the left and b_μ or b_ν to the right of any term in which they occur. Next we may pull all of the E_t''s past the H_t's in $\xi(u_\mu)\xi_t'(v_\nu)$ and use the relations (4.5) to obtain

$$\begin{aligned} u_\mu v_\nu &\equiv \xi_t(u_\mu)\left(\frac{t}{n}E, \frac{t}{n}H\right) \\ &\cdot \xi_t'(v_\nu)\left(\frac{n-t}{n}E, \frac{n-t}{n}H\right) \bmod \mathfrak{P}'U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}' \end{aligned} \tag{4.11}$$

where the product on the right-hand side is taken in the polynomial ring $\mathbf{C}[E] \otimes \mathbf{C}[H] \simeq \mathbf{C}[E, H]$.

Finally we observe that, by (1.24) and (1.25),

$$\xi_t(u_\mu)\left(\frac{t}{n}E, \frac{t}{n}H\right) = \xi(u_\mu)(E, H) \tag{4.12}$$

and

$$\xi_t'(v_\nu)\left(\frac{n-t}{n}E, \frac{n-t}{n}H\right) = \xi(v_\nu)(E, H). \tag{4.13}$$

This completes the proof of Proposition 1.2.

REMARK. It is not clear that Proposition 1.2 remains valid for an arbitrary tube domain since the subalgebras $\mathfrak{g}_t$ and $\mathfrak{g}_t'$ need not behave so simply as in our case.

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